On Shokurov's rational connectedness conjecture - Part II

Main theorem:

Let (X,Δ) be a log pair, and let $f: X \to S$ be a projective morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X+\Delta))$ is relatively generated for some $m>0$. Let $g: Y \to X$ be any birational morphism, and let $\pi: Y \to S$ be the composite morphism.
Then the connected components of every fiber of π are rationally chain connected modulo the inverse image of the locus of log canonical singularities of (X,Δ).

Proof:

- Replace $f: X \to S$ with the Stein Factorization.
- Let $s \in S$ closed point, consider an affine neighborhood of s,

we can assume that $-K_X$ is big and $-(K_X+\Delta)$ is semi ample.

Claim: We may assume that $\Delta = A+B$ with A ample, B effective, $K_X+\Delta \sim_{\mathbb{Q}} 0$.

Pick $D \in |-m(K_X+\Delta)|$ general. Then $K_X+\Delta+D/m$ has the same non-klt locus as $K_X+\Delta$.
Then $\Delta + D/m \sim_{\mathbb{Q}} -K_X$, and so is big. $\Delta \sim_{\mathbb{Q}} A+B$, A ample B effective.
$\Delta' = (1-\varepsilon)\Delta + \varepsilon A + \varepsilon B$. For $\varepsilon \ll 1$ the non-klt locus of (X,Δ') is contained in the non-klt locus of (X,Δ).

- Because g and f have connected fibers, we can replace g by $Y' \xrightarrow{g'} Y \xrightarrow{g} X$, g' birational.

- Consider $\begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend right, "\pi"'] & X \arrow[r, "f"] & S \end{tikzcd}$ as follows:
  - * take G effective $\mathbb{Q}$-Cartier divisor on S
  - * Let $g: Y \to X$ s.t. the fiber $\pi^{-1}(s)$ union the exceptional locus of g union the strict transform of $f^* G + \Delta$ has SNC support.
  - * Let F be the union of the components of the fiber $\pi^{-1}(s)$ which have log discrepancy greater than 0.

Theorem:
We may pick effective $\mathbb{Q}$-divisors G on S and Γ and E on Y with the following properties.

1) The equation

$$K_Y + \Gamma \sim_{\mathbb{Q},\pi} E$$

holds, where Γ and E have no common components, E g-exceptional, and $\Gamma = A + B$, with A π-ample and B eff.

Proof:

- Write $K_Y + \Gamma' = g^*(K_X+\Delta) + E'$, with Γ', E' with no common components, $g_*\Gamma' = \Delta$, E' g-exceptional
- Let E^* the sum of all the exceptional divisors with coeff 1. Set $\Gamma'' = \Gamma' + \delta E^*$, $E'' = E' + \delta E^*$, with δ sufficiently small positive rational number.

Γ'' is π-big, so $\Gamma'' \sim_{\mathbb{Q},\pi} A + B$, with A π-ample, A and E'' have no common components, B eff.

Consider

$$K_Y + (1-\varepsilon)\Gamma'' + \varepsilon A + \varepsilon B \sim_{\mathbb{Q},\pi} g^*(K_X+\Delta) + E''$$

Cancel common terms

$$K_Y + \Gamma \sim_{\mathbb{Q}} g^*(K_X+\Delta) + E$$

And so $K_Y + \Gamma \sim_{\mathbb{Q},\pi} E$.

(2) Given $t \in [0,1]$, set $\Delta_t = \Delta + t f^* G$ and Γ_t, E_t be the eff. divisors obtained by cancelling common comp of $\Gamma + t\pi^* G$ and E.
Let V_t be the closure of

$$\mathrm{nkl}(Y, \Gamma_t) - \mathrm{nkl}(Y, \Gamma)$$

Then $F = V_1$.

Proof:
Choose G sufficiently singular at s, so $F \subset V_1$. Choose H sufficiently ample. so $\mathcal{O}_S(H) \otimes m_s^{\otimes l}$ is g.g.

(3) We may assume that there are rational numbers $0 = t_0 < t_1 < \ldots < t_k = 1$ s.t $V_{t_i} = F_1 \cup \ldots \cup F_i$, where the F_i are the irreducible components of F.

Proof:
For every $0 \leq t \leq 1$, V_t is the union of some components of F.
Let t_i be the smallest value s.t. F_i appears as a component of V_{t_i}.
By a result by Ambro '98, we can perturb G to get $t_i \neq t_j$ if $i \neq j$.

(4) Denote Θ_i the restriction of $\{\Gamma_{t_i}\}$ to F_i. Let H be any π-ample divisor. Then there is a constant M such that

Fractional part

$$h^0(F_i, \mathcal{O}_{F_i}(m(K_{F_i} + \Theta_i) + H|_{F_i})) \leq M$$

for all $m \gg 0$.

Lemma:

Let $F \subset Y$ be a smooth divisor in a smooth variety, and let $\pi: Y \to S$ be a projective morphism. Let H be a suff. π-very ample divisor, and set $A = (\dim X + 1)H$. Assume

(1) Γ is a $\mathbb{Q}$-divisor with SNC support s.t. Γ contains F with coeff 1 and (Y, Γ) is log canonical,

(2) $C \geq 0$ is a $\mathbb{Q}$-divisor not containing F,

(3) $K_F + \Theta$ is π-pseudoeffective, where $\Theta = (\Gamma - F)|_F$,

(4) There is a divisor $G \geq 0$, $G \sim K_Y + \Gamma + C$ which does not contain any log canonical center of $(Y, \lceil \Gamma \rceil)$.

Then for $m \gg 0$, the image of
$$\pi_* \mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A) \to \pi_* \mathcal{O}_F(m(K_F + \Theta + C|_F) + H|_F + A|_F)$$
contains the image of the sheaf
$$\pi_* \mathcal{O}_F(m(K_F + \Theta) + H|_F).$$

Proof of (4)

$\Gamma = \{\Gamma_{t_i}\} + F_i$; $C = \Gamma_{t_i} - \{\Gamma_{t_i}\} - F_i \geq 0$

$H = H$; $A = (\dim X + 1) H$

$F = F_i$; $\Theta = \Theta_i$; $G = E_{t_i}$

Assume $K_{F_i} + \Theta_i$ is psef.
By the lemma, we can lift sections of

$$\underset{\text{//}}{H^0(m(K_{F_i} + \Theta_i) + H|_{F_i})}$$
$\pi_* \mathcal{O}_{F_i}(m(K_{F_i} + \Theta_i) + H|_{F_i})$ to
$\pi_* \mathcal{O}_Y(m(K_Y + \Gamma_{t_i}) + (\dim X + 2)H)$

Pick D on X s.t. $g^* D \geq (\dim X + 2) H$
Then for $m \gg 0$ $m(K_Y + \Gamma_{t_i}) = m E_{t_i}$
$$\pi_* \mathcal{O}_Y(m(K_Y + \Gamma_{t_i}) + (\dim X + 2)H) \subseteq \pi_*(m E_{t_i} + g^* D) = f_* \mathcal{O}_X(D)$$

(5) F_i is rationally chain connected modulo $W_i = F_i \cap \mathrm{nklt}(Y, \Gamma_{t_{i-1}})$

Proof:
Let $\alpha: F_i \dashrightarrow Z$ any rational map such that the non-klt locus of $K_{F_i} + (\Gamma_{t_i} - F_i)|_{F_i}$ does not dominate Z.
We need to check three conditions:

(1) $K_{F_i} + (\Gamma_{t_i} - F_i)|_{F_i}$ has Kodaira dim at least zero on the general fiber of α.
(2) $K_{F_i} + (\Gamma_{t_i} - F_i)|_{F_i}$ has Kodaira dim at most zero.
(3) There is an ample divisor A on F_{t_i} s.t. $A \leq \Gamma_{t_i} - F_i$

(3) ✓ by construction
(2) Property (4) implies that the sections of $m(K_{F_i} + \Theta_i) + H|_{F_i}$ are bounded.

(1) Recall that $K_Y + \Gamma_{t_i} \sim_{\mathbb{Q},\pi} E_{t_i} \geq 0$, and $F_i \not\subseteq \mathrm{Supp}(E_{t_i})$. Then the Kodaira dimension of $K_Y + \Gamma_{t_i}$ restricted to the general fiber is at least zero.

Because W_i does not dominate Z, we have $(\Gamma_{t_i} - F_i)|_{F_i} = \{\underset{\substack{\vee| \\ 0}}{F_i}\}|_{F_i}$ on the general fiber.

So $K_{F_i} + (\Gamma_{t_i} - F_i)|_{F_i}$ has Kodaira dimension at least 0 on the general fiber.